

# Adverse Selection in Competitive Search Equilibrium

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## **Abstract**

We extend the notion of competitive search equilibrium to an environment with adverse selection. Uninformed principals post contracts to attract informed agents. Agents observe the contracts and apply for one, trading off the probability of matching with a principal against the terms of trade offered by the contract. We characterize equilibria as the solution to a constrained optimization problem and show that in equilibrium principals offers separating contracts to attract different types of agents. We then present a set of examples, including a workplace rat race, insurance against layoff risk, and lemons in asset markets, to illustrate the usefulness of our model.

# 1 Introduction

This paper studies equilibrium and efficiency in an economy with adverse selection. A large number of uninformed principals compete to attract a large number of informed agents. We extend the notion of competitive search equilibrium to allow for private information. Principals post incentive-compatible contracts which specify an action profile if they match with a particular type of agent. Agents observe the posted contracts and direct their search towards the most attractive ones. Matching is limited by the restriction that each principal can match with at most one agent, and also possibly by search frictions. For example, if fewer principals offer a particular contract than the number of agents who wish to obtain it, each agent matches only probabilistically. Principals and agents form rational expectations about the market tightness of each contract—the ratio of principals posting that contract to agents who direct their search to that contract—as well as the composition of agents who search for the contract.

Part of the contribution of this paper is technical: we develop a canonical extension to the competitive search model (Montgomery, 1991; Peters, 1991; Moen, 1997; Shimer, 1996; Acemoglu and Shimer, 1999; Burdett, Shi and Wright, 2001; Mortensen and Wright, 2002) that allows for ex-ante heterogeneous agents with private information about their type. We prove that, under mild assumptions, including a weak version of a single-crossing condition, there exists an equilibrium where principals offer separating contracts: each contract posted attracts only one type of agent, and different types of agents direct their search towards different types of contracts. The expected utility of each type of agent is uniquely determined in equilibrium. Moreover, the set of competitive search equilibria is easily found by sequentially solving a constrained optimization problem for each type of agent.

We also present a series of examples and applications of our model. These serve three purposes: they illustrate the usefulness of our approach; they show that some well-known results in either contract theory or search theory can change when we combine elements of both in one model; and they enable us to explore the role of our assumptions and discuss what happens when they are relaxed.

The first example is a version of the classic costly signaling model (Spence, 1973; Akerlof, 1976). Workers are heterogeneous both in their productivity and in their cost of working under unpleasant conditions, for example long workdays. More productive workers find long workdays to be less costly. We focus on an extreme case where firms care about workers' productivity but not about the length of the workday. In equilibrium, firms separate workers by making more productive workers work longer days, a version of a rat race. Curiously, we find that the probability that each type of worker gets a job—the unemployment rate—

is unchanged from a version of the model with symmetric information, and hence with an efficient workday. We prove that a Pareto improvement is feasible if there are sufficiently few unproductive agents or if the difference in the cost of working long days is small relative to the difference in productivity levels. We also consider a version of this example where long workdays are cheaper for less productive workers. In this case there exists an equilibrium where workers are pooled and there is no distortion in the length of the workday, but we find that other equilibrium may also exist.

Our second example is a modified version of the Rothschild and Stiglitz (1976) insurance model.<sup>1</sup> Risk-averse workers and risk-neutral firms match in pairs in order to produce output. However, only some pairs are productive and workers differ in the probability that they can form a productive match. In equilibrium, firms separate workers by partially insuring them against the productivity shock. In particular, workers are worse off if they lose their job than if they had never found a job. We interpret this example as providing an explanation for why firms do not insure workers against the risk of a layoff. To do so would risk attracting too many unproductive workers. We show that, even if a pooling contract does not Pareto dominate the equilibrium, a partial pooling allocation—only pooling some types of workers—may be Pareto superior.

This example shows that competitive search offers a resolution to the famous nonexistence problem in Rothschild and Stiglitz (1976). When there are relatively few low productivity workers, equilibrium may not exist in the original Rothschild-Stiglitz model, because any separating contract is less profitable than a pooling contract that cross-subsidizes low productivity workers. In our model, such a pooling contract is infeasible, regardless of the composition of the worker pool. The key difference is that in our model, firms are small and so a deviation cannot attract the entire population. Suppose a firm posts a contract that is supposed to attract a representative cross-section of the population. Because it can match with at most one worker, the more workers who try to obtain a contract, the less likely each worker is to match. This drives some workers away from the contract. Critically, the most productive workers are the first to leave, because their outside option—trying to obtain a separating contract—is more attractive. This means that only undesirable workers would be attracted by such a deviation, which makes it unprofitable.

In the first two examples, our use of competitive search equilibrium affects the contracts that are offered in equilibrium but asymmetric information does not affect the equilibrium

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<sup>1</sup>This example builds on a discussion between Daron Acemoglu and one of the authors. We discuss the relationship between this example and the original Rothschild-Stiglitz model more in the body of the text. We focus on a labor market application because we find the assumption that a firm can only hire a small fraction of the available workers more reasonable than the assumption that an insurance company can only cover a small fraction of the population.

frictions. Our third example reverses this: asymmetric information makes it harder for some agents to find a trading partner but does not affect the terms of trade conditional on finding a partner. We study a stylized model of asset trade (Akerlof, 1970). Agents want to sell a heterogeneous object, say apples that could be good or bad, to principals. Principals value apples more than agents do, so there are gains from trade. However, bad apples are valued less than good apples—they are lemons. There are no search frictions, so everyone on the short side of the market will match.

We show that in equilibrium agents holding good apples only trade probabilistically. The probability of a meeting, rather than the terms of trade within a meeting, screens out the agents holding bad apples. This economizes on any cost that principals must incur in meeting an agent. Once again, we find that the competitive search equilibrium is Pareto dominated by a pooling allocation if there are few enough agents holding bad apples. We also find that if there are no gains from trade in bad apples, adverse selection will entirely shut down the market for good apples, an extreme version of the lemons problem.

Our paper is related to a growing literature exploring search models with private information. In particular, Faig and Jerez (2005), Guerrieri (2008), and Moen and Rosén (2006) propose different extensions of competitive search models with one-side private information. However, in all these papers agents are *ex ante* homogeneous and all heterogeneity is match-specific. Inderst and Müller (1999) is an exception that extends the standard notion of competitive search to an environment with *ex ante* heterogeneous agents. That model is a (slightly) special case of our first example. Inderst and Wambach (2001) explore a version of the Rothschild and Stiglitz (1976) model with a finite number of principals and agents and a capacity constraint for each agent. This paper is related to our second example. Our paper is the first to develop a general framework for defining and analyzing competitive search with adverse selection in a wide variety of applications.

Many papers have studied related economies with adverse selection but without search. Driven by the nonexistence issue in Rothschild and Stiglitz (1976), Miyazaki (1977), Wilson (1977), and Riley (1979) propose alternative notions of equilibrium that offer possible resolutions. In contrast to these papers, we generally find that there is no cross-subsidization in equilibrium. The key difference is our assumption that each principal is small relative to the number of agents. Since he cannot attract all the agents with a pooling contract, a principal must deduce which agents are most attracted to such a contract, which we show eliminates the incentive to pool.

Prescott and Townsend (1984) study adverse selection in competitive economies, concluding pessimistically that “there do seem to be fundamental problems for the operation of competitive markets for economies or situations which suffer from adverse selection.” (p. 44)

More recently, Bisin and Gottardi (2006) propose a notion of Walrasian equilibrium with competitive markets, but agents are restricted to trade only incentive-compatible contracts. They also find that there always exists a separating equilibrium. Although our notion of equilibrium is more strategic, their equilibrium allocation has some features that are similar. In particular, the incentive-compatibility condition that they impose on the set of admissible trades is analogous to a condition that arises endogenously in our model.

The rest of the paper is organized as follows. In Section 2, we develop the general environment, define competitive search equilibrium, and discuss the critical assumptions. In Section 3, we show how to find competitive search equilibria by solving a relatively simple constrained optimization problem. We prove that a separating equilibrium always exists and show that the equilibrium vector of agents' payoffs is unique. In Section 4, we define the class of incentive-feasible allocations, and discuss whether equilibrium outcomes are efficient within this class. In Sections 5-7, we explore the examples and applications discussed above; in each case we characterize equilibria and discuss efficiency. Section 8 concludes.

## 2 Model

We consider a static model. There is a measure 1 of agents, a fraction  $\pi_i > 0$  of whom are of type  $i \in \mathbb{I} \equiv \{1, 2, \dots, I\}$ . The type is the agent's private information. There is also a large measure of ex ante homogeneous principals. Principals and agents have a single opportunity to match.

A principal may post a contract, at cost  $k > 0$ , which gives him an opportunity to match with an agent. We discuss the nature of contracts in the next paragraph. Let  $\mathbb{Y} \subset \mathbb{R}^n$  denote the space of feasible action profiles for principals and agents who are matched, and assume  $\mathbb{Y}$  is compact and nonempty. A typical action profile  $y \in \mathbb{Y}$  may specify actions by the principal, actions by the agent, and transfers between them, among other possibilities. A principal who matches with a type  $i$  agent gets a payoff  $v_i(y) - k$  if they undertake the action profile  $y \in \mathbb{Y}$ . A principal who does not post a contract gets payoff normalized to zero, while one who posts a contract but fails to match gets  $-k$ . A type  $i$  agent matched with a principal gets payoff  $u_i(y)$  if they undertake the action profile  $y \in \mathbb{Y}$ , while unmatched agents get a payoff normalized to zero. Assume  $u_i : \mathbb{Y} \mapsto \mathbb{R}$  and  $v_i : \mathbb{Y} \mapsto \mathbb{R}$  are continuous and bounded for all  $i$ .

We use the revelation principle to assume without loss of generality that the contracts are revelation mechanisms. More precisely, a contract is a vector of action profiles,  $C \equiv \{y_1, \dots, y_i, \dots, y_I\} \in \mathbb{Y}^I$ , specifying that if a principal and agent match, the latter announces her type  $i$ , and they implement  $y_i$ . A contract  $C \equiv \{y_1, \dots, y_I\}$  is incentive compatible if

$u_i(y_i) \geq u_i(y_j)$  for all  $i$ .<sup>2</sup> Let  $\mathbb{C} \subset \mathbb{Y}^I$  denote the set of incentive compatible contracts. Principals only post incentive compatible contracts.

We turn now to the matching process. We assume that all agents observe the set of posted contracts and direct their search to the most attractive ones. Let  $\Theta(C)$  denote the ratio of principals offering contract  $C \in \mathbb{C}$  to agents who direct their search towards that contract,  $\Theta : \mathbb{C} \mapsto [0, \infty]$ . Let  $p_i(C)$  denote the share of these agents whose type is  $i$ , with  $P(C) \equiv \{p_1(C), \dots, p_i(C), \dots, p_I(C)\} \in \Delta^I$ , the  $I$ -dimensional unit simplex. That is,  $P(C)$  satisfies  $p_i(C) \geq 0$  for all  $i$  and  $\sum_i p_i(C) = 1$  and so  $P : \mathbb{C} \mapsto \Delta^I$ . The functions  $\Theta$  and  $P$  are determined endogenously in equilibrium and are defined for all incentive compatible contracts, not only the ones that are posted in equilibrium.

A type  $i$  agent seeking contract  $C$  matches with a principal with probability  $\mu(\Theta(C))$ , independent of her type, where  $\mu : [0, \infty] \mapsto [0, 1]$  is nondecreasing. A principal offering contract  $C$  matches with a type  $i$  agent with probability  $\eta(\Theta(C))p_i(C)$ , where  $\eta : [0, \infty] \mapsto [0, 1]$  is nonincreasing. We impose that  $\mu(\theta) = \theta\eta(\theta)$  for all  $\theta$  since the left hand side is the matching probability of an agent and the right hand side is the matching probability of a principal times the principal-agent ratio. Together with the monotonicity of  $\mu$  and  $\eta$ , this implies both functions are continuous. It is convenient to let  $\bar{\eta} \equiv \eta(0) > 0$  denote the highest probability that a principal can match with an agent, obtained when the principal-agent ratio for a contract is 0. Similarly let  $\bar{\mu} \equiv \mu(\infty) > 0$  denote the highest probability that an agent can match with a principal.

We summarize the setup by writing the expected utilities of principals and agents. The expected utility of a principal who posts  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$  is

$$\eta(\Theta(C)) \sum_{i=1}^I p_i(C) v_i(y_i) - k.$$

The expected utility of a type  $i$  agent who seeks contract  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$  and reports type  $j$  is

$$\mu(\Theta(C)) u_i(y_j).$$

Incentive compatibility says that the agent is willing to report truthfully,  $u_i(y_i) \geq u_i(y_j)$ .

We now generalize the notion of competitive search equilibrium to our environment with ex ante heterogeneous agents and asymmetric information.

**Definition 1** *A competitive search equilibrium is a vector  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}} \in \mathbb{R}_+^I$ , a measure  $\lambda$  on  $\mathbb{C}$  with support  $\bar{\mathbb{C}}$ , a function  $\Theta(C) : \mathbb{C} \mapsto [0, \infty]$ , and a function  $P(C) : \mathbb{C} \mapsto \Delta^I$*

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<sup>2</sup>Note that we are not concerned with moral hazard in this paper and so we assume that an action profile  $y \in \mathbb{Y}$  can be implemented by any principal and agent  $i$ .

satisfying

(i) **principals' profit maximization and free-entry:** for any  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$ ,

$$\eta(\Theta(C)) \sum_{i=1}^I p_i(C) v_i(y_i) \leq k,$$

with equality if  $C \in \bar{\mathbb{C}}$ ;

(ii) **agents' optimal search:** let

$$\bar{U}_i = \max \left\{ 0, \max_{C' = \{y'_1, \dots, y'_I\} \in \bar{\mathbb{C}}} \mu(\Theta(C')) u_i(y'_i) \right\};$$

then for any  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$  and  $i$ ,

$$\bar{U}_i \geq \mu(\Theta(C)) u_i(y_i),$$

with equality if  $\Theta(C) < \infty$  and  $p_i(C) > 0$ ; moreover, if  $u_i(y_i) < 0$ , either  $\Theta(C) = \infty$  or  $p_i(C) = 0$ ;

(iii) **market clearing:**

$$\int_{\bar{\mathbb{C}}} \frac{p_i(C)}{\Theta(C)} d\lambda(\{C\}) \leq \pi_i \text{ for any } i,$$

with equality if  $\bar{U}_i > 0$ .

In equilibrium, principals and agents take as given the expected utility of each type of agent,  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ . Notice that  $\bar{U}_i \geq 0$  for all  $i$ , as all agents can choose not to participate and obtain their outside option 0. Moreover, principals and agents have rational expectations about the market tightness and the distribution of agents' types associated with each contract,  $\Theta$  and  $P$ .

Given these expectations, principals post contracts to maximize their expected profits. Free entry ensures that profits are non-positive, and equal to zero for contracts that are posted in equilibrium. The restriction that  $\eta(\Theta(C)) \sum_{i=1}^I p_i(C) v_i(y_i) \leq k$  captures the idea that if some contract  $C = \{y_1, \dots, y_I\}$  offered positive profits, more principals would post it, driving down  $\eta(\Theta(C))$ .

Also given their expectations, agents search optimally for contracts. The expected utility of a type  $i$  agent is the highest utility that he can obtain from any contract that is posted in equilibrium, or zero—the value of the option not to search—if there are no contracts posted in equilibrium or if any equilibrium contract offers him negative utility. If an arbitrary contract

$C$  offers type  $i$  agents less than  $\bar{U}_i$ , they will not direct their search towards that contract, nor will they direct their search towards a contract that gives negative utility should the agent succeed in obtaining it.<sup>3</sup> The restriction that  $\bar{U}_i \geq \mu(\Theta(C))u_i(y_i)$  captures the idea that if type  $i$  agents anticipated that they could obtain higher utility from contract  $C = \{y_1, \dots, y_I\}$ , more would direct their search towards that contract, driving down  $\mu(\Theta(C))$ .

Finally, the market clearing condition guarantees that all type  $i$  agents direct their search towards some contract, unless they are indifferent between matching and their outside option,  $\bar{U}_i = 0$ .

Define

$$\bar{\mathbb{Y}}_i \equiv \{y \in \mathbb{Y} \mid \bar{\eta}v_i(y) \geq k \text{ and } u_i(y) \geq 0\},$$

the set of action profiles for type  $i$  agents that deliver nonnegative utility to the agent while permitting the principal to make nonnegative profits if the principal-agent ratio is equal to zero. Also define

$$\bar{\mathbb{Y}} \equiv \bigcup_i \bar{\mathbb{Y}}_i.$$

In equilibrium, action profiles that are not in  $\bar{\mathbb{Y}}$  are not implemented. For much of the analysis, we make three assumptions on preferences over action profiles  $y \in \bar{\mathbb{Y}}$ , which we now present and discuss.

**Assumption A1** *Monotonicity: for all  $y \in \bar{\mathbb{Y}}$ ,*

$$v_1(y) \leq v_2(y) \leq \dots \leq v_I(y).$$

This says that, for any fixed action profile, principals weakly prefer higher types. Although this is more than a normalization, the assumption could be relaxed at the cost of notation.

**Assumption A2** *Local non-satiation: for all  $i \in \mathbb{I}$ ,  $j < i$ ,  $y \in \bar{\mathbb{Y}}_i$ , and  $\varepsilon > 0$ , there exists a  $y' \in B_\varepsilon(y)$  such that  $v_i(y') > v_i(y)$  and  $u_j(y') \leq u_j(y)$ .*

Here  $B_\varepsilon(y) \equiv \{y' \in \mathbb{Y} \mid d(y, y') < \varepsilon\}$  and  $d(y, y')$  is the Euclidean distance between the two points, so  $B_\varepsilon$  is a ball of radius  $\varepsilon$ . This is again a mild assumption, immediately satisfied in any example where the action profile  $y$  allows transfers. All of our results go through even without this assumption if we restrict attention to strictly monotonic matching functions.

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<sup>3</sup>The restriction that if  $u_i(y_i) < 0$ , either  $\Theta(C) = \infty$  or  $p_i(C) = 0$  rules out the possibility that type  $i$  agents search for contract  $C = \{y_1, \dots, y_I\}$  only because they know that they have no chance of obtaining the contract,  $\Theta(C) = 0$ , and so get zero utility from this activity. Allowing for this possibility may enlarge the set of equilibria by permitting off-equilibrium beliefs that seem unreasonable to us.



**Assumption A3** *Sorting: for all  $i \in \mathbb{I}$ ,  $y \in \bar{\mathbb{Y}}_i$ , and  $\varepsilon > 0$ , there exists a  $y' \in B_\varepsilon(y)$  such that*

$$u_j(y') > u_j(y) \text{ for all } j \geq i \text{ and } u_j(y') < u_j(y) \text{ for all } j < i.$$

This is an important assumption, guaranteeing that it is possible for principals to design contracts that attract desirable but not undesirable agents. This is a generalized version of a standard single-crossing condition. That is, assume that  $y \equiv (y_1, y_2)$  has two dimensions, that we can ignore boundaries in the action profiles, e.g. because boundary points of  $\mathbb{Y}$  are not elements of  $\bar{\mathbb{Y}}$ , and that agents' utility functions are differentiable. Then A3 holds if the marginal rate of substitution between  $y_1$  and  $y_2$  is higher for higher types,

$$\frac{\partial u_i(y_1, y_2) / \partial y_1}{\partial u_i(y_1, y_2) / \partial y_2}$$

is monotone in  $i$ . Still, A3 is more general. For example, suppose

$$u_i(y_1, y_2) = \left(\frac{1}{2}y_1^{\rho_i} + \frac{1}{2}y_2^{\rho_i}\right)^{1/\rho_i} - \frac{y_1^2 + y_2^2}{2} - \frac{1}{8}.$$

for some  $\rho_i \leq 1$ . Also assume  $\rho_i$  is higher for higher types. Note that the elasticity of substitution between  $y_1$  and  $y_2$  is  $1/(1 - \rho_i)$ , increasing in  $i$ . Let  $\mathbb{Y} = [0, 1]^2$ , a superset of the points where  $u_i(y_1, y_2) > 0$ . Indeed, by construction any point  $(y, 1, y_2)$  on the boundary of  $\mathbb{Y}$  is not in  $\bar{\mathbb{Y}}$  since  $u_i(y_1, y_2) < 0$ . It is easy to verify that this example fails the standard single-crossing condition at action profiles of the form  $(y, y)$ . However, it satisfies A3 since it is always possible to increase the spread between  $y_1$  and  $y_2$ , attracting higher types and repelling lower types. In Section 5.4 we consider an example where A3 fails, and show that this can substantially change the nature of equilibrium.

### 3 Characterization

We characterize the equilibrium as the solution to a set of optimization problems. For any type  $i$ , consider the following problem (P- $i$ ):

$$\begin{aligned} & \max_{\theta \in [0, \infty], y \in \mathbb{Y}} \mu(\theta) u_i(y) && \text{(P-}i\text{)} \\ \text{s.t. } & \eta(\theta) v_i(y) \geq k, \\ & \text{and } \mu(\theta) u_j(y) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

We say that a set  $\mathbb{I}^* \subset \mathbb{I}$  and three vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve problem (P) if:

1.  $\mathbb{I}^*$  denotes the set of  $i$  such that the constraint set of problem (P- $i$ ) is non empty and the maximized value is strictly positive, given  $(\bar{U}_1, \dots, \bar{U}_{i-1})$ ;
2. for any  $i \in \mathbb{I}^*$ , the pair  $(\theta_i, y_i)$  solves problem (P- $i$ ) given  $(\bar{U}_1, \dots, \bar{U}_{i-1})$  and  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$
3. for any  $i \notin \mathbb{I}^*$ ,  $\bar{U}_i = 0$ .

Our main result is that we can find any equilibrium by solving problem (P) and conversely that any solution to problem (P) is an equilibrium. We first prove that problem (P) has a solution and provide a partial characterization of the solution.

**Lemma 1** *Assume A1-A3. There exists a set  $\mathbb{I}^*$  and vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  that solve problem (P). At a solution,*

$$\begin{aligned} \eta(\theta_i)v_i(y_i) &= k \text{ for all } i \in \mathbb{I}^*, \\ \mu(\theta_i)u_j(y_i) &\leq \bar{U}_j \text{ for all } j \in \mathbb{I} \text{ and } i \in \mathbb{I}^*, \end{aligned}$$

**Proof.** In the first step, we prove that there exists a solution to (P). The second and third steps establish the two properties.

**Step 1** Let us begin with  $i = 1$ . If the constraint set of problem (P-1) is empty, we simply set  $\bar{U}_1 = 0$ . If the constraint set is non empty, then problem (P-1) is well-behaved: the objective function is continuous in  $(\theta, y)$  because  $\mu$  and  $u_1$  are continuous, the set of  $(\theta, y)$  satisfying the constraint  $\eta(\theta)v_1(y) \geq k$  is closed because  $\eta$  and  $v_1$  are continuous, and, since  $[0, \infty] \times \mathbb{Y}$  is compact, the constraint set is compact. Hence, (P-1) has a solution and a unique maximum  $\bar{U}_1$ . If  $\bar{U}_1 > 0$ , let  $(\theta_1, y_1)$  be one of the maximizers.

We now proceed by induction. Fix  $i > 1$  and assume that we have found  $\bar{U}_j$  for all  $j < i$  and  $(\theta_j, y_j)$  for all  $j \in \mathbb{I}^*$ ,  $j < i$ . We now focus on problem (P- $i$ ). If the constraint set is empty, we again set  $\bar{U}_i = 0$ . If instead the constraint set is non-empty, then problem (P- $i$ ) is well-behaved: the objective function is continuous in  $(\theta, y)$  because  $\mu$  and  $u_i$  are continuous functions, the set of  $(\theta, y)$  satisfying the constraints is compact because  $\eta$ ,  $\mu$ ,  $v_i$ , and  $u_j$  for any  $j < i$  are all continuous functions, and  $y \in \mathbb{Y}$  which is compact by assumption. Hence, (P- $i$ ) has a solution and a unique maximum  $\bar{U}_i$ . If  $\bar{U}_i > 0$ , let  $(\theta_i, y_i)$  be one of the maximizers.

**Step 2** Suppose by way of contradiction that there exists  $i \in \mathbb{I}^*$  such that  $(\theta_i, y_i)$  solves (P- $i$ ), but  $\eta(\theta_i)v_i(y_i) > k$ . This together with the fact that  $\bar{U}_i = \mu(\theta_i)u_i(y_i) > 0$ , implies that  $y_i \in \bar{Y}_i$  and  $\mu(\theta_i) > 0$ . Fix  $\varepsilon > 0$  such that for all  $y \in B_\varepsilon(y_i)$ ,  $\eta(\theta_i)v_i(y) \geq k$ . Then, assumption A3 ensures there exists a  $y' \in B_\varepsilon(y_i)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_i) \text{ for all } j \geq i, \\ u_j(y') &< u_j(y_i) \text{ for all } j < i. \end{aligned}$$

Then the pair  $(\theta_i, y')$  satisfies all the constraints of problem (P- $i$ ):

1.  $\eta(\theta_i)v_i(y') \geq k$ , from the choice of  $\varepsilon$ ;
2.  $\mu(\theta_i)u_j(y') < \mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  for all  $j < i$ , where the first inequality comes from the construction of  $y'$  and from  $\mu(\theta_i) > 0$ , and the second inequality from the assumption that  $(\theta_i, y_i)$  solves (P- $i$ ).

Moreover, the pair  $(\theta_i, y')$  achieves a higher value of the objective function in problem (P- $i$ ) than does  $(\theta_i, y_i)$ ,  $\mu(\theta_i)u_i(y') > \mu(\theta_i)u_i(y_i)$ ; again the inequality comes from the construction of  $y'$  and from  $\mu(\theta_i) > 0$ . Hence,  $(\theta_i, y_i)$  does not solve (P- $i$ ), a contradiction. This proves that  $\eta(\theta_i)v_i(y_i) = k$  for all  $i \in \mathbb{I}^*$ .

**Step 3** Fix  $i \in \mathbb{I}^*$  and suppose by way of contradiction that there exists  $j > i$  such that  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ . Let  $h$  be the smallest such  $j$ .

Note that since  $i \in \mathbb{I}^*$ ,  $\mu(\theta_i)u_i(y_i) = \bar{U}_i > 0$ , which implies that  $\mu(\theta_i) > 0$  and  $u_i(y_i) > 0$ . Also, since  $(\theta_i, y_i)$  solves (P- $i$ ), it satisfies the constraint  $\eta(\theta_i)v_i(y_i) \geq k$ , which ensures  $\eta(\theta_i) > 0$  and  $v_i(y_i) > 0$ . In particular,  $y_i \in \bar{Y}_i$ .

Next, the pair  $(\theta_i, y_i)$  satisfies the constraints of problem (P- $h$ ) since

1.  $\eta(\theta_i)v_h(y_i) \geq \eta(\theta_i)v_i(y_i) \geq k$ , where the first inequality holds by assumption A1 because  $h > i$  and  $y_i \in \bar{Y}_i \subset \bar{Y}$ , and the second holds because  $(\theta_i, y_i)$  solves (P- $i$ );
2.  $\mu(\theta_i)u_l(y_i) \leq \bar{U}_l$  for all  $l < h$ , which holds for
  - (a)  $l < i$  because  $(\theta_i, y_i)$  satisfy the constraints of (P- $i$ ),
  - (b)  $l = i$  because  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$  by problem (P) and  $i \in \mathbb{I}^*$ ,
  - (c)  $i < l < h$  by the choice of  $h$  as the smallest violation of  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ .

Since by assumption,  $\mu(\theta_i)u_h(y_i) > \bar{U}_h$ , we have not solved problem (P-h), a contradiction. This proves that  $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  for all  $j \in \mathbb{I}$  and  $i \in \mathbb{I}^*$ . ■

The next proposition proves that we can use any solution to problem (P) to construct an equilibrium.

**Proposition 1** *Assume A1-A3. Consider a set  $\mathbb{I}^*$  and three vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  that solve problem (P). There exists a competitive search equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$  with  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\bar{\mathbb{C}} = \{C_i\}_{i \in \mathbb{I}^*}$  where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $p_i(C_i) = 1$ .*

**Proof.** We proceed by construction.

- The vector of expected utilities is  $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ .
- The set of posted contracts is  $\bar{\mathbb{C}} = \{C_i\}_{i \in \mathbb{I}^*}$  where  $C_i \equiv (y_i, \dots, y_i)$ .
- $\lambda$  is such that  $\lambda(\{C_i\}) = \pi_i \Theta(C_i)$  for any  $i \in \mathbb{I}^*$ .
- For  $i \in \mathbb{I}^*$  and  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ . Otherwise, for any incentive compatible contract  $C' = \{y'_1, \dots, y'_I\} \in \mathbb{C}$ , let  $J(C') = \{j | u_j(y'_j) > 0\}$  denote the types that attain positive utility from the contract. If  $J(C') \neq \emptyset$  and  $\min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\} < \bar{\mu}$  then

$$\mu(\Theta(C')) = \min_{j \in J(C')} \frac{\bar{U}_j}{u_j(y'_j)}.$$

If this equation is consistent with multiple values of  $\Theta(C')$ , as may happen if  $\mu$  is not strictly increasing, pick one, e.g. the smallest such value for  $\Theta(C')$ . Otherwise, if  $J(C') = \emptyset$  or  $\min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\} \geq \bar{\mu}$ , then  $\Theta(C') = \infty$ .

- For  $i \in \mathbb{I}^*$  and  $C_i = (y_i, \dots, y_i)$ , let  $p_i(C_i) = 1$  and so  $p_j(C_i) = 0$  for  $j \neq i$ . For any other  $C'$ , define  $P(C')$  such that  $p_h(C') > 0$  only if  $h \in \arg \min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\}$ . If there are multiple elements of the arg min, pick one such value for  $P(C')$ , e.g.  $p_h(C') = 1$  if  $h$  is the smallest element of the arg min. If  $J(C') = \emptyset$ , again choose  $P(C')$  arbitrarily, e.g. set  $p_1(C') = 1$ .

**Condition (i)** For any  $i \in \mathbb{I}^*$ , the pair  $(\theta_i, y_i)$  solves problem (P-i) and in particular Lemma 1 implies  $\eta(\theta_i)v_i(y_i) = k$ . This implies that profit maximization and free entry hold for any posted contract  $\{C_i\}_{i \in \mathbb{I}^*}$ .

Now consider an arbitrary incentive compatible contract; we show that principals' profit maximization and free-entry condition is satisfied. Suppose, to the contrary, that there exists an incentive compatible contract  $C' = (y'_1, \dots, y'_I) \in \mathbb{C}$  with  $\eta(\Theta(C')) \sum_i p_i(C')v_i(y'_i) > k$ .

Note that this implies  $\eta(\Theta(C')) > 0$  and so  $\Theta(C') < \infty$ . In particular there exists some type  $j$  with  $p_j(C') > 0$  and  $\eta(\Theta(C'))v_j(y'_j) > k$ . Since  $p_j(C') > 0$  and  $\Theta(C') < \infty$ , our construction of  $\Theta(C')$  implies  $\bar{U}_j = \mu(\Theta(C'))u_j(y'_j)$ . Similarly, for all  $h$ ,

$$\bar{U}_h \geq \mu(\Theta(C'))u_h(y'_h) \geq \mu(\Theta(C'))u_h(y'_j),$$

where the first inequality follows from the construction of  $\Theta$  and the second from the requirement that  $C'$  is incentive compatible. The preceding inequalities prove that  $(\Theta(C'), y'_j)$  satisfies the constraints of problem (P- $j$ ). Since  $\eta(\Theta(C'))v_j(y'_j) > k$ , Lemma 1 implies that there exists some pair  $(\theta'', y''_j)$  that satisfies the constraints of problem (P- $j$ ) but delivers a higher value of the objective function,  $\mu(\theta'')u_j(y''_j) > \bar{U}_j \geq 0$ . Since  $(\theta'', y''_j)$  is in the constraint set of problem (P- $j$ ) and it delivers a strictly positive value for the objective,  $j \in \mathbb{I}^*$ . But then the fact that  $\bar{U}_j$  is not the maximized value of (P- $j$ ) is a contradiction.

**Condition (ii)** By construction, the equilibrium functions  $\Theta$  and  $P$  ensure that  $\bar{U}_i \geq \mu(\Theta(C'))u_i(y'_i)$  for all contracts  $C' = \{y'_1, \dots, y'_I\}$ , with equality if  $\Theta(C') < \infty$  and  $p_i(C') > 0$ . Moreover, problem (P) ensures that, for any  $i \in \mathbb{I}^*$ ,  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$  where  $\theta_i = \Theta(C_i)$  and  $C_i = \{y_i, \dots, y_i\}$  is the equilibrium contract offered to type  $i$  agents.

**Condition (iii)** The market clearing condition is satisfied by the construction of  $\lambda$ . ■

The next proposition establishes the converse, that any equilibrium can be characterized using problem (P).

**Proposition 2** *Assume A1-A3. Let  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$  be a competitive search equilibrium. Let  $\{\bar{U}_i\}_{i \in \mathbb{I}} = \bar{U}$  and  $\mathbb{I}^* = \{i \in \mathbb{I} | \bar{U}_i > 0\}$ . For each  $i \in \mathbb{I}^*$ , there exists a contract  $C \in \bar{\mathbb{C}}$  with  $\Theta(C) < \infty$  and  $p_i(C) > 0$ . Moreover, take any  $\{\theta_i\}_{i \in \mathbb{I}^*}$  and  $\{y_i\}_{i \in \mathbb{I}^*}$  such that for each  $i \in \mathbb{I}^*$ , there exists a contract  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  (so the  $i^{\text{th}}$  element of  $C_i$  is  $y_i$ ) with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ . Then the set  $\mathbb{I}^*$  and vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve problem (P).*

**Proof.** From part (i) of the definition of equilibrium, any  $C \in \bar{\mathbb{C}}$  has  $\eta(\Theta(C)) > 0$ , hence  $\Theta(C) < \infty$ . From part (iii) of the definition,  $\bar{U}_i > 0$  implies  $p_i(C) > 0$  for some  $C \in \bar{\mathbb{C}}$ . This proves that for each  $i \in \mathbb{I}^*$ , there exists a contract  $C \in \bar{\mathbb{C}}$  with  $\Theta(C) < \infty$  and  $p_i(C) > 0$ .

The remainder of the proof proceeds in five steps. The first four steps prove that for any  $i \in \mathbb{I}^*$  and  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ ,  $(\theta_i, y_i)$  solves problem (P- $i$ ). First, we prove that the constraint  $\eta(\theta_i)v_i(y_i) \geq k$  is satisfied. Second, we prove that the constraint  $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  is satisfied for all  $j$ . Third, we prove that such a

pair  $(\theta_i, y_i)$  actually delivers  $\bar{U}_i$  to type  $i$ . Finally, we prove that  $(\theta_i, y_i)$  solves problem (P- $i$ ). The fifth step proves that for any  $i \notin \mathbb{I}^*$ , either the constraint set of problem (P- $i$ ) is empty or the maximized value is non-positive.

**Step 1** Take  $i \in \mathbb{I}^*$  and  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ . We prove that the constraint  $\eta(\theta_i)v_i(y_i) \geq k$  is satisfied in problem (P- $i$ ). Note first that  $i \in \mathbb{I}^*$  implies  $\bar{U}_i > 0$ . Since part (ii) of the definition of equilibrium implies  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$ , this ensures  $\mu(\theta_i) > 0$ .

To find a contradiction, now assume  $\eta(\theta_i)v_i(y_i) < k$ . Part (i) of the definition of equilibrium implies  $\eta(\theta_i) \sum_j p_j(C)v_j(y_j) = k$ , and so there is an  $h$  with  $p_h(C) > 0$  and  $\eta(\theta_i)v_h(y_h) > k$ . Since  $\eta(\theta_i) \leq \bar{\eta}$ ,  $\bar{\eta}v_h(y_h) > k$  as well. Moreover, because  $\theta_i = \Theta(C) < \infty$  and  $p_h(C) > 0$ , optimal search implies  $u_h(y_h) \geq 0$ . This proves that  $y_h \in \bar{\mathbb{Y}}_h$ .

Next, fix  $\varepsilon > 0$  such that for all  $y \in B_\varepsilon(y_h)$ ,  $\eta(\theta_i)v_h(y) > k$ . Then assumption A3 together with  $y_h \in \bar{\mathbb{Y}}$  guarantees that there exists  $y' \in B_\varepsilon(y_h)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_h) \text{ for all } j \geq h, \\ u_j(y') &< u_j(y_h) \text{ for all } j < h. \end{aligned}$$

Notice that  $y' \in \bar{\mathbb{Y}}_h$  as well, given that  $u_h(y') > u_h(y_h) \geq 0$  and  $\bar{\eta}v_h(y') \geq \eta(\theta_i)v_h(y') > k$ .

Next consider the contract  $C' = \{y', \dots, y'\}$ . Let  $\theta' \equiv \Theta(C')$ . Note that

$$\mu(\theta')u_h(y') \leq \bar{U}_h = \mu(\theta_i)u_h(y_h) < \mu(\theta_i)u_h(y'),$$

where the first inequality uses the optimal search condition for contract  $C'$ , the second equality holds by optimal search because  $\theta_i < \infty$  and  $p_h(C) > 0$ , and the last inequality holds by the construction of  $y'$ , since  $\mu(\theta_i) > 0$ . This implies  $\mu(\theta') < \mu(\theta_i)$ ; since  $\mu$  is nondecreasing, it also implies  $\theta' < \theta_i$ .

Next observe that for all  $j < h$ , either  $u_j(y') < 0$ , in which case  $p_j(C') = 0$  by part (ii) of the definition of equilibrium, or

$$\mu(\theta')u_j(y') < \mu(\theta_i)u_j(y_h) \leq \mu(\theta_i)u_j(u_j) \leq \bar{U}_j,$$

where the first inequality uses  $\mu(\theta') < \mu(\theta_i)$  and the construction of  $y'$ , the second uses incentive compatibility of  $C$ , and the third inequality follows from the optimal search condition for contract  $C$ . This implies  $p_j(C') = 0$  for all  $j < h$ .

Finally, the profits from posting contract  $C'$  are

$$\eta(\theta') \sum_{j=1}^I p_j(C') v_j(y') \geq \eta(\theta') v_h(y') \geq \eta(\theta_i) v_h(y') > k.$$

The first inequality follows because  $p_j(C') = 0$  if  $j < h$  and  $v_h(y')$  is nondecreasing in  $h$  by assumption A1 together with  $y' \in \bar{Y}_h \subset \bar{Y}$ . The second follows because  $\theta' < \theta_i$  and  $\eta$  is nonincreasing. The last inequality uses the construction of  $\varepsilon$ . Offering the contract  $C'$  is therefore strictly profitable, a contradiction.

**Step 2** Again take  $i \in \mathbb{I}^*$  and  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ . Part (ii) of the definition of equilibrium implies  $\mu(\theta_i) u_j(y_j) \leq \bar{U}_j$  for all  $j$  while incentive compatibility implies  $u_j(y_i) \leq u_j(y_j)$ . This proves that the constraint  $\mu(\theta_i) u_j(y_i) \leq \bar{U}_j$  in problem (P- $i$ ) is satisfied for all  $j$ .

**Step 3** Again take  $i \in \mathbb{I}^*$  and  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ . Part (ii) of the definition of equilibrium implies  $\bar{U}_i = \mu(\theta_i) u_i(y_i)$ , since  $\theta_i < \infty$  and  $p_i(C_i) > 0$ . This proves that  $(\theta_i, y_i)$  delivers utility  $\bar{U}_i$  to type  $i$  agents.

**Step 4** Once again take  $i \in \mathbb{I}^*$  and  $C_i = \{y_1, \dots, y_i, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $\theta_i = \Theta(C_i) < \infty$  and  $p_i(C_i) > 0$ . To find a contradiction, suppose there exists  $(\theta', y')$  that satisfies the constraints of problem (P- $i$ ) but delivers higher utility. That is,  $\eta(\theta') v_i(y') \geq k$ ,  $\mu(\theta') u_j(y') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta') u_i(y') > \bar{U}_i$ .

We now use assumption A2. Note that  $\mu(\theta') u_i(y') > \bar{U}_i > 0$  implies  $\mu(\theta') > 0$  and  $u_i(y') > 0$ , while  $\eta(\theta') v_i(y') \geq k > 0$  implies  $\eta(\theta') > 0$  and  $v_i(y') > 0$ . In particular,  $y' \in \bar{Y}_i$ . We can therefore fix  $\varepsilon > 0$  such that for all  $y \in B_\varepsilon(y')$ ,  $\mu(\theta') u_i(y) > \bar{U}_i$ , and then choose  $y'' \in B_\varepsilon(y')$  such that  $v_i(y'') > v_i(y')$  and  $u_j(y'') \leq u_j(y')$  for all  $j < i$ . In particular, this ensures  $\eta(\theta') v_i(y'') > k$ ,  $\mu(\theta') u_j(y'') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta') u_i(y'') > \bar{U}_i$ . Note that we still have  $y'' \in \bar{Y}_i$ .

We next use assumption A3. Fix  $\varepsilon' > 0$  such that for all  $y \in B_{\varepsilon'}(y'')$ ,  $\eta(\theta') v_i(y) > k$  and  $\mu(\theta') u_i(y'') > \bar{U}_i$ . Choose  $y''' \in B_{\varepsilon'}(y'')$  such that

$$\begin{aligned} u_j(y''') &> u_j(y'') \text{ for all } j \geq i, \\ u_j(y''') &< u_j(y'') \text{ for all } j < i. \end{aligned}$$

In particular, this ensures  $\eta(\theta') v_i(y''') > k$ ,  $\mu(\theta') u_j(y''') < \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta') u_i(y''') > \bar{U}_i$ . Finally, note that we still have  $y''' \in \bar{Y}_i$ .

Now consider the contract  $C''' \equiv \{y''', \dots, y'''\}$ . From part (ii) of the definition of equilibrium,  $\mu(\theta')u_i(y''') > \bar{U}_i$  implies that  $\mu(\theta') > \mu(\Theta(C'''))$ , which in turn guarantees that  $\eta(\Theta(C'''))v_i(y''') > k$ .

We next claim that  $p_j(y''') = 0$  for all  $j < i$ . First suppose  $\mu(\Theta(C''')) = 0$  and  $p_j(y''') > 0$  for some  $j < i$ . This implies  $\bar{U}_j = 0$ . But since  $\mu(\theta') > 0$ ,  $\mu(\theta')u_j(y'') \leq \bar{U}_j$  implies  $u_j(y'') \leq 0$ , while by construction  $u_j(y''') < u_j(y'')$ . Part (ii) of the definition of equilibrium therefore implies that either  $\Theta(C''') = \infty$  or  $p_j(y''') = 0$ , a contradiction. So instead assume  $\mu(\Theta(C''')) > 0$ . Recall  $\mu(\theta')u_j(y'') \leq \bar{U}_j$ ,  $\mu(\Theta(C''')) < \mu(\theta')$ , and  $u_j(y''') < u_j(y'')$ . These inequalities jointly imply  $\mu(\Theta(C'''))u_j(y''') < \bar{U}_j$ , hence  $p_j(y''') = 0$ .

The profit from offering this contract is therefore

$$\eta(\Theta(C''')) \sum_{j=1}^I p_j(C''') v_j(y''') \geq \eta(\Theta(C''')) v_i(y''') > k,$$

where the first inequality uses  $p_j(C''') = 0$  for  $j < i$  and  $v_j(y''')$  is increasing in  $j$ , by assumption A1. This is a contradiction, which proves  $(\theta_i, y_i)$  solves (P- $i$ ).

**Step 5** Suppose there is an  $i \notin \mathbb{I}^*$  for which the constraint set of problem (P- $i$ ) is nonempty and the maximized value is positive. That is, there exists a  $(\theta_i, y_i)$  such that  $\eta(\theta')v_i(y') \geq k$ ,  $\mu(\theta')u_j(y') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y') > \bar{U}_i = 0$ . Replicating step 4, we can first find a  $y''$  such that  $\eta(\theta')v_i(y'') > k$ ,  $\mu(\theta')u_j(y'') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y'') > 0$ . Then we find a  $y'''$  such that  $\eta(\theta')v_i(y''') > k$ ,  $u_j(y''') > u_j(y'')$  for  $j < i$ , and  $u_i(y''') < u_i(y'')$ . Finally, prove that the contract  $C''' = \{y''', \dots, y'''\}$  only attracts type  $i$  or higher agents and so must be profitable and must deliver them positive utility, a contradiction. ■

It is worth stressing that we use assumption A2 in the proof of Proposition 2, but nowhere else in the paper. We use the assumption because we need to establish that it is possible to make the principal better off while not improving the well-being of agents. If  $\eta$  is strictly decreasing, we could do this by reducing  $\eta$ ; however, for our examples in Sections 5–7, it is convenient to allow that  $\eta$  is only weakly decreasing, and so we introduce this additional assumption.

The next proposition combines the previous results to prove existence of equilibrium and uniqueness of equilibrium payoffs.

**Proposition 3** *Assume A1-A3 hold. Then competitive search equilibrium exists. Moreover, the equilibrium vector  $\bar{U}$  is unique.*



**Proof.** Lemma 1 shows that, under A1-A3, there exists a solution for problem (P). Moreover, Proposition 1 shows that, under the same assumptions, if the set  $\mathbb{I}^*$  and the vectors  $\{\bar{U}_i\}_{i \in \mathbb{I}}$ ,  $\{\theta_i\}_{i \in \mathbb{I}^*}$ , and  $\{y_i\}_{i \in \mathbb{I}^*}$  solve problem (P), there exists a competitive search equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$  with the same  $\bar{U}$ ,  $\bar{\mathbb{C}} = \{C_i\}_{i \in \mathbb{I}^*}$ , where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $p_i(C_i) = 1$ . This proves existence.

Proposition 2 shows that in any competitive search equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$ ,  $\bar{U}_i$  is the maximum value of the objective of problem (P- $i$ ) for all  $i \in \mathbb{I}^*$  and  $\bar{U}_i = 0$  otherwise. Also, Lemma 1 shows that there exists a unique maximum value  $\bar{U}_i$  for the objective of problem (P- $i$ ) for all  $i \in \mathbb{I}^*$ . This proves uniqueness. ■

When there are strict gains from trade for all types, we can prove that all types get positive utility. We show by example in Section 7 below that the possibility of strict gains from trade for type  $i$  is not enough to ensure positive utility for type  $i$  agents.

**Proposition 4** *Assume A1-A3 hold and that for all  $i$ , there exists  $y \in \mathbb{Y}$  with  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Then in any competitive search equilibrium,  $\bar{U}_i > 0$  for all  $i$  and in particular there exists a contract  $C \in \mathbb{C}$  with  $\Theta(C) < \infty$  and  $p_i(C) > 0$ .*

**Proof.** We prove  $\bar{U}_i > 0$  for all  $i$  using problem (P). Start with  $i = 1$ . Fix  $y$  satisfying  $\bar{\eta}v_1(y) > k$  and  $u_1(y) > 0$ . Then fix  $\theta > 0$  satisfying  $\eta(\theta)v_1(y) = k$ . These points satisfy the constraints of problem (P-1) and deliver utility  $\mu(\theta)u_1(y) > 0$ . This proves  $\bar{U}_1 > 0$ .

Now suppose we have proven that  $\bar{U}_j > 0$  for all  $j < i$ . We prove  $\bar{U}_i > 0$ . Again fix  $y$  satisfying  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Then fix  $\theta > 0$  satisfying  $\eta(\theta)v_i(y) \geq k$  and  $\mu(\theta)u_j(y) \leq \bar{U}_j$  for all  $j < i$ ; this is feasible since  $\bar{U}_j > 0$  and  $\mu$  is continuous with  $\mu(0) = 0$ . These points satisfy the constraints of problem (P- $i$ ) and deliver utility  $\mu(\theta)u_i(y) > 0$ , which proves  $\bar{U}_i > 0$  and establishes the induction step. ■

## 4 Incentive Feasible Allocations

This section sets the stage for studying the efficiency properties of equilibrium. In particular, we define an incentive feasible allocation, which is necessary to look for Pareto improvements. An allocation is a vector of expected utilities for the different types of agents, a set of posted contracts, and the associated market tightness and composition of agents applying for the posted contracts.

**Definition 2** *An allocation is a vector  $\bar{U}$  of expected utilities for the agents, a measure  $\lambda$  over the set of incentive-compatible contracts  $\mathbb{C}$  with support  $\bar{\mathbb{C}}$ , a function  $\tilde{\Theta} : \bar{\mathbb{C}} \mapsto [0, \infty]$ ,*

and a function  $\tilde{P} : \bar{\mathbb{C}} \mapsto \Delta^I$ .<sup>4</sup>

It is incentive feasible whenever: (1) each posted contract offers the maximal expected utility to agents who direct their search for that contract and no more to those who do not; (2) the economy's resource constraint is satisfied; and (3) markets clear.

**Definition 3** *An allocation  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \tilde{\Theta}, \tilde{P}\}$  is incentive feasible if*

1. *for any  $C \in \bar{\mathbb{C}}$  and  $i \in \{1, \dots, I\}$  such that  $\tilde{p}_i(C) > 0$  and  $\tilde{\Theta}(C) < \infty$ ,*

$$\bar{U}_i = \mu(\tilde{\Theta}(C))u_i(y_i),$$

*and*

$$\bar{U}_i \equiv \max_{C' \in \bar{\mathbb{C}}} \mu(\tilde{\Theta}(C'))u_i(y'_i)$$

*where  $C' = \{y'_1, \dots, y'_I\}$ ;*

- 2.

$$\int \left( \eta(\tilde{\Theta}(C)) \sum_{i=1}^I \tilde{p}_i v_i(y_i) - k \right) d\lambda(C) = 0;$$

3. *for all  $i \in \{1, \dots, I\}$ ,*

$$\int \frac{\tilde{p}_i(C)}{\tilde{\Theta}(C)} d\lambda(C) \leq \pi_i,$$

*with equality if  $\bar{U}_i > 0$ .*

The set of incentive feasible allocations provides a good benchmark for what the economy may be able to achieve through legal restrictions on the type of contracts that can be offered.

## 5 Rat Race

We now proceed through three examples that illustrate the usefulness of our model.

### 5.1 Setup

Our first example shows that in competitive search equilibrium principals can separate agents by distorting directly the terms of the posted contracts in a version of a classic signaling

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<sup>4</sup>Notice that  $\tilde{\Theta}$  and  $\tilde{P}$  are different objects than  $\Theta$  and  $P$  because they are defined only over the set of posted contracts.

model (Akerlof, 1976).<sup>5</sup> We think of agents as workers who are heterogeneous both in terms of their productivity and their cost of working long hours. Principals can be thought of as firms that are willing to pay more to hire more productive workers, but cannot observe productivity, only hours worked. We show that if the cost of a long workday is lower for more productive workers, firms will use hours worked to separate workers.

An action profile here consists of two elements,  $y = \{t, x\}$ , where  $t$  denotes a transfer from the firm to the worker and  $x \geq 0$  denotes the length of the workday. The payoff of a matched worker of type  $i$  who undertakes action  $\{t, x\}$  is

$$u_i(t, x) = t - \frac{x}{a_i},$$

where higher values of  $a_i$  imply that  $x$  is less costly. The payoff of a firm matched with a type  $i$  agent who takes action  $\{t, x\}$  is

$$v_i(t, x) = b_i - t,$$

where  $b_i$  is the productivity of the worker.

Assume  $\mu$  is strictly concave and continuously differentiable. Also assume that  $I = 2$  and, without loss of generality, that type 2 workers are more productive than type 1 workers:  $b_2 > b_1$ . We restrict the set of feasible action profiles to  $\mathbb{Y} = \{(t, x) | t \in [-\varepsilon, b_2] \text{ and } x \in [0, b_2 \max\{a_1, a_2\}]\}$  for some number  $\varepsilon > 0$ . We view the restriction that  $x \geq 0$  as technological, while the other restrictions ensure  $\mathbb{Y}$  is compact but are otherwise without loss of generality. A firm would never profit from offering a transfer higher than  $b_2$  and the worker would never accept a negative transfer, although it is convenient to allow for the possibility of a slightly negative transfer. Moreover, a worker would never provide set  $x > b_2 \max\{a_1, a_2\}$ , given that the transfer is bounded by  $b_2$ ; she would prefer not to search.

The space of action profiles that provide nonnegative utility to a type  $i$  worker and nonnegative profit to a firm when the firm-worker ratio is zero are

$$\bar{\mathbb{Y}}_i = \{(t, x) \in \mathbb{Y} | x/a_i \leq t \leq b_i - k/\bar{\eta}\}.$$

The fact that  $b_2 > b_1$  immediately implies that assumption A1 is satisfied. Assumption A2 holds because  $(t, x) \in \bar{\mathbb{Y}}_i$  implies  $t \geq 0$  and so can be reduced to raise  $v_i(t, x)$  and lower  $u_j(t, x)$ ; for this reason it is convenient to allow negative transfers.

The critical assumption is A3. Consider points  $(t, 0) \in \bar{\mathbb{Y}}$ . There are nearby points  $(t', x')$

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<sup>5</sup>If we set  $a_i = b_i$ , this example is equivalent to a static version of Inderst and Müller (1999), although the definitions of equilibrium are conceptually different.

with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$  if and only if  $a_2 > a_1$ , so the more productive worker finds it less costly to set higher  $x$ . At any other point  $\{t, x\} \in \bar{\mathbb{Y}}$ , assumption A3 holds for all values of  $a_1$  and  $a_2$ . We therefore impose  $a_2 > a_1$  in what follows and discuss the nature of equilibrium if this assumption is violated at the end of this section.

Finally, we assume that  $\bar{\eta}b_1 > k$  so that there are gains from trade for both types of workers. By proposition 4, this implies that the equilibrium is characterized by both  $\bar{U}_1 > 0$  and  $\bar{U}_2 > 0$ .

## 5.2 Competitive Search Equilibrium

Using the results in Section 3, we can characterize a competitive search equilibrium using vectors  $(\bar{U}_1, \bar{U}_2)$ ,  $(t_1, x_1, t_2, x_2)$ , and  $(\theta_1, \theta_2)$  that solve problem (P). In this example, problem (P-i) is

$$\begin{aligned} \bar{U}_i &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_i} \right) \\ \text{s.t. } &\eta(\theta)(b_i - t) \geq k, \\ \text{and } &\mu(\theta) \left( t - \frac{x}{a_j} \right) \leq \bar{U}_j \text{ for } j \leq i. \end{aligned}$$

We claim the following result:

**Result 1** *There exists a unique competitive search equilibrium with  $\mu'(\theta_i)b_i = k$ , for  $i = 1, 2$ , and so  $\theta_1 < \theta_2$ ;  $x_1 = 0$ ,*

$$x_2 = \frac{a_1}{\mu(\theta_2)} \left[ \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) - \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) \right] k > 0;$$

and

$$t_i = \left( \frac{1}{\mu'(\theta_i)} - \frac{\theta_i}{\mu(\theta_i)} \right) k$$

for  $i = 1, 2$ . Moreover,

$$\bar{U}_1 = \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k \text{ and } \bar{U}_2 = \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) \right] k.$$

**Proof.** We prove this result by first solving problem (P-1), finding  $\bar{U}_1$  and then using it to solve problem (P-2).

Consider problem (P-1). Using  $\eta(\theta) = \mu(\theta)/\theta$ , we write it as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_1} \right) \text{ s.t. } \frac{\mu(\theta)}{\theta} (b_1 - t) \geq k.$$

Lemma 1 implies that the constraint is binding so that we can eliminate  $t$  and reduce the problem to

$$\bar{U}_1 = \max_{\theta \in [0, \infty], x \in [0, b_2 a_2]} \mu(\theta) \left( b_1 - \frac{x}{a_1} \right) - \theta k.$$

It is easy to see that at the optimum  $x = 0$  and  $\theta = \theta_1$ , where  $\theta_1$  solves the necessary and sufficient first order condition,  $\mu'(\theta_1)b_1 = k$ . Using this to eliminate  $b_1$  from the objective function delivers the expression for  $\bar{U}_1$ . Also use the constraint to solve for  $t_1$ .

Next, solve problem (P-2) using the solution for  $\bar{U}_1$ , that is,

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_2} \right) \\ \text{s.t. } &\frac{\mu(\theta)}{\theta} (b_2 - t) \geq k, \\ \text{and } &\mu(\theta) \left( t - \frac{x}{a_1} \right) \leq \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k. \end{aligned}$$

Again using Lemma 1, the first constraint binds and so we can eliminate  $t$ . It is easy to verify that the second constraint must be binding as well, so we can use it to eliminate  $x$  and then check that at the solution  $x \geq 0$ . The problem reduces to

$$\bar{U}_2 = \max_{\theta \in [0, \infty]} \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k + \left( 1 - \frac{a_1}{a_2} \right) (\mu(\theta)b_2 - \theta k) \right].$$

Then, at the solution  $\theta = \theta_2$ , where  $\theta_2$  solves the necessary and sufficient first order condition,  $\mu'(\theta_2)b_2 = k$ . Note that concavity of  $\mu$  implies  $\theta_1 < \theta_2$ . Substituting into the objective function gives the expression for  $\bar{U}_2$ . Finally, use the constraints to compute  $t_2$  and  $x_2$ . Again, concavity of  $\mu$  ensures that  $\mu(\theta)/\mu'(\theta) - \theta$  is increasing in  $\theta$ , which in turn implies  $x_2 > 0$ . ■

In equilibrium, some firms post contracts to attract only type 1 workers and others post contracts to attract only type 2 workers. In order to separate the two types of workers, firms have to offer distorted contracts to workers of type 2, requiring a costly signal, while the contracts offered to the type 1 workers are undistorted. Two features of this example merit mention. First, the market tightness margin is equal to the first-best (symmetric information) level for both types of contracts. This is because the worker bears the full cost

of the distortion. Second, without additional assumptions, we cannot rule out the possibility that the transfer to type 2 workers is lower than the transfer to type 1 workers. This is in addition to the cost  $x > 0$  that they must incur. Their reward for applying to firms offering type 2 contracts is only a higher probability of trade,  $\theta_2^* > \theta_1^*$ .

### 5.3 Pareto Optimality

For some parameter values, the competitive search equilibrium is Pareto inefficient. Consider an allocation that treats the two types of workers identically,  $\bar{C} = \{C\}$ , where  $C = ((t, 0), (t, 0))$ . Moreover, assume  $\tilde{\Theta}(C) = \theta^*$ , with  $\theta^*$  solving  $\mu'(\theta^*)(\pi_1 b_1 + \pi_2 b_2) = k$ ,  $\tilde{p}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1/\theta^*$ . Note that this defines  $\theta_1 < \theta^* < \theta_2$ , an intermediate level of market tightness. Finally, choose  $t$  such that firms make zero profit, that is,

$$t = \left( \frac{1}{\mu'(\theta^*)} - \frac{\theta^*}{\mu(\theta^*)} \right) k$$

The contract is incentive compatible since the types are treated identically. Moreover, all workers are attracted to the posted contracts and get the same expected utility, so condition (1) of feasibility is satisfied. Also, the resource constraint and the market clearing condition are satisfied by the choice of  $t$  and  $\lambda$ . Hence, it constitutes an incentive feasible allocation.

The expected utility of (both types of) workers is now

$$\bar{U} = \left( \frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \right) k.$$

Compare this with the equilibrium. Since  $\theta^* > \theta_1$ , trivially  $\bar{U} > \bar{U}_1$ . On the other hand,  $\bar{U} \geq \bar{U}_2$  if and only if

$$\frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \geq \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right).$$

This holds if  $a_1/a_2$  is sufficiently close to 1 (screening is very costly) or if  $\pi_1$  is sufficiently close to zero (there are few type 1 workers). The reason is that in equilibrium, firms who want to attract type 2 need to screen out type 1 agents. If a firm failed to do so, it would be swamped by type 1 workers. This may not be optimal, however. If there are few type 1 workers or screening is too costly, it is preferable to subsidize type 1 workers and eliminate costly screening.

## 5.4 Pooling

The sorting condition A3 plays a substantive role in the analysis of our model. To understand the role of such a condition, consider now a variant of the same example where signaling is cheaper for the less productive workers,  $a_1 \geq a_2$  while  $b_1 < b_2$ . As we show below, this violates assumption A3. Firms would like to screen out low productivity workers, but the only available screening technology works against them. If a firm posts a contract with the desire to attract only the more productive workers, the less productive ones would report to be more productive. We show that in this case there is a class of equilibria in which firms attract both types of workers and may prescribe the less productive workers to set  $x \geq 0$ . The most efficient (and arguably most natural) of these equilibria is the one where firms offer the same action profiles to all the workers and allow  $x = 0$ . Notice that, in this case, to characterize the equilibrium we cannot use the analysis in Section 3, which relies on assumption A3, but we need to go back to the primitive Definition 1 of a competitive search equilibrium.

The restriction that  $a_1 \geq a_2$  does not affect assumptions A1 and A2, but it violates assumption A3. Fix  $t$  and set  $x = 0$ . For any nearby contract  $(t', x')$ ,

$$u_1(t', x') - u_1(t, 0) = t' - t - x'/a_1 \geq t' - t - x'/a_2 = u_2(t', x') - u_2(t, 0),$$

since  $x' \geq 0$ . It follows that there is no such value of  $(t', x')$  with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$ .

We claim that in this case, there exists a class of equilibria indexed by the amount of action  $x$  required from low productivity agents,  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ . In equilibrium, all firms post the same contract  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ , where  $t$  is chosen to ensure that firms make zero profits. Moreover, given that all the posted contracts are the same, all types of workers look for the same contracts, and  $p_i(C) = \pi_i$ . A firm might consider offering a contract that attracts only one type of worker. If she tries to attract only type 1 workers, she will lose the benefit of cross-subsidization and thus is unable to attract them while earning positive profits. If she tries to attract type 2 workers, she will be unable to devise a contract that will exclude type 1 workers, again making such a deviation unprofitable.

**Result 2** *Suppose  $a_1 \geq a_2$ . For any  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ , there exists a competitive search equilibrium where  $\bar{C} = \{C\}$  with  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$  and*

$$t = \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 - \frac{\theta}{\mu(\theta)} k,$$

where  $\theta$  solves

$$\mu'(\theta) \left( \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 \right) = k.$$

Moreover, the expected utility of both types of workers is

$$\bar{U} = \mu(\theta)t.$$

**Proof.** Fix  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$ . Our proof proceeds by constructing an equilibrium. Assume that  $\bar{\mathcal{C}} = \{C\}$  where  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ ,  $\bar{U}_1 = \bar{U}_2 = \bar{U} = \mu(\theta)t$ , and  $\lambda(\{C\}) = 1/\theta$ , where  $t$  and  $\theta$  are defined above. Moreover,  $\Theta(C) = \theta$  and  $p_i(C) = \pi_i$  for  $i = 1, 2$ . For any other incentive compatible contract  $C' = \{(t'_1, x'_1), (t'_2, x'_2)\} \in \bar{\mathcal{C}}$ ,  $C' \neq C$ , suppose  $\Theta(C')$  solves

$$\bar{U}_1 = \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right)$$

if this defines  $\Theta(C') < \infty$ ; if  $\bar{U} \geq \bar{\mu}(t'_1 - x'_1/a_1)$ ,  $\Theta(C') = \infty$ . Finally, suppose  $p_1(C') = 1$  and  $p_2(C') = 0$  for all such contracts.

By construction, profit maximization and free entry hold for the posted contract  $C$ . In particular,  $t$  is chosen so that firms break even.

For any other incentive-compatible contract  $C' \neq C$ , the firm's profit maximization and free entry condition reduces to

$$\eta(\Theta(C'))(b_1 - t'_1) \leq k.$$

Since  $\eta(\infty) = 0$ , this is obviously satisfied for contracts with  $\Theta(C') = \infty$ . Otherwise, use the construction of  $\Theta(C')$  to eliminate  $t'_1$  from this requirement. We need to show that

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \bar{U}_1.$$

An upper bound on the left hand side is obtained by setting  $x'_1 = 0$  and choosing  $\Theta(C')$  to maximize the left hand side,  $\mu'(\Theta(C'))b_1 = k$ . The restriction  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$  implies that  $b_1 \leq \pi_1(b_1 - \frac{x_1}{a_1}) + \pi_2 b_2$ , from which it follows that  $\Theta(C') \leq \theta$ . That is,

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \left( \frac{\mu(\Theta(C'))}{\mu'(\Theta(C'))} - \Theta(C') \right) k \leq \left( \frac{\mu(\theta)}{\mu'(\theta)} - \theta \right) k = \bar{U}_1,$$

where the first inequality uses the preceding discussion, the second inequality holds because  $\Theta(C') \leq \theta$ , and the third holds from the construction of  $\bar{U}_1$ .

Next, workers' optimal search for an incentive-compatible contract  $C'$  holds by construc-



tion for type 1 workers. For type 2 workers, we need to verify that

$$\bar{U}_2 \geq \mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right).$$

To prove this, note that  $t'_1 - x'_1/a_1 \geq t'_2 - x'_2/a_1 \geq t'_2 - x'_2/a_2$ , where the first inequality comes from incentive compatibility of  $C'$  and the second from the assumption  $a_1 \geq a_2$  and the feasibility restriction  $x'_2 \geq 0$ . This implies that

$$\mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right) \leq \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right) = \bar{U}_1,$$

which establishes the desired inequality since  $\bar{U}_1 = \bar{U}_2$ .

Finally, the market clearing condition holds by construction. ■

This result shows that there exists a class of Competitive Search Equilibria parameterized by  $x_1$ , the extent to which low productivity workers are forced to use the costly action. In this example, firms would like to be able to screen the more productive type 2 workers, but the assumption  $a_1 \leq a_2$  implies that they can use the action  $x$  only to attract less productive type 1 workers. Still, there are equilibria where  $x$  is positive because firms fear that if they did not require  $x > 0$  from type 1 workers, they would be stuck exclusively with that type of worker.

A positive value of  $x$  is socially wasteful, so these equilibria can be Pareto ranked. In particular, the “pooling” equilibrium characterized by  $x_1 = 0$  is Pareto optimal, at least within this class. In such an equilibrium, all the firms offer the same contract that prescribes the same action profile to all the workers:  $x = 0$  and a transfer which ensures that the firms break even.

Next, we show that when  $a_1$  is strictly larger than  $a_2$ , then any competitive search equilibrium where all the firms post the same contract falls in the class of equilibria characterized in Result 2.

**Result 3** *Suppose  $a_1 > a_2$ . If there exists a competitive search equilibrium with  $\bar{C} = \{C\}$  where  $C = ((t_1, x_1), (t_2, x_2))$ ,  $p_i(C) = \pi_i$ , and  $\Theta(C) < \infty$ , then  $x_2 = 0$ , and  $t_1 - x_1/a_1 = t_2$ .*

**Proof.** Throughout this proof, we suppose there is an equilibrium characterized by the single incentive compatible contract  $C = ((t_1, x_1), (t_2, x_2))$ . In the first step we prove that  $x_2 = 0$  and in the second that  $t_1 - x_1/a_1 = t_2$ .

**Step 1** Suppose  $x_2 > 0$ . Given that  $p_i(C) = \pi_i > 0$  for  $i = 1, 2$  and  $\Theta(C) < \infty$ , optimal search requires that the expected utility of type  $i$  workers satisfies

$$\bar{U}_i = \mu(\Theta(C)) \left( t_i - \frac{x_i}{a_i} \right),$$

for  $i = 1, 2$ . Next, consider the contract  $C' = ((t_2 - x_2/a_2, 0), (t_2 - x_2/a_2, 0))$ . Optimal search of type 2 workers requires that

$$\mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right) \leq \bar{U}_2 = \mu(\Theta(C)) \left( t_2 - \frac{x_2}{a_2} \right),$$

which implies that  $\Theta(C') \leq \Theta(C)$ . Moreover, notice that

$$t_1 - \frac{x_1}{a_1} \geq t_2 - \frac{x_2}{a_1} > t_2 - \frac{x_2}{a_2},$$

where the first inequality follows from incentive compatibility of  $C$ , and the second from the fact that  $a_1 > a_2$  and  $x_2 > 0$ . This, together with  $\Theta(C') \leq \Theta(C)$ , implies that

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right).$$

It follows that type 1 workers will never search for  $C'$ , that is,  $p_1(C') = 0$ . Hence, given that  $\Theta(C') \leq \Theta(C) < \infty$ , it must be that  $p_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . Then, the expected profits for a firm posting  $C'$  are

$$\eta(\Theta(C')) \left( b_2 - t_2 + \frac{x_2}{a_2} \right) > \eta(\Theta(C)) (\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k.$$

The first inequality follows from  $b_1 < b_2$ ; the fact shown above that  $t_1 > t_2 - x_2/a_2 + x_1/a_1 \geq t_2 - x_2/a_2$ ; and  $\Theta(C') = \Theta(C)$ . The second equality follows from the fact that in the proposed equilibrium firms post  $C$  and break even. Hence, contract  $C'$  represents a profitable deviation, a contradiction.

**Step 2** We now prove that  $t_1 - x_1/a_1 = t_2$ . Notice that incentive compatibility of  $C$  and the result from the previous step that  $x_2 = 0$  imply that  $t_1 - x_1/a_1 \geq t_2$ . To derive a contradiction, suppose that  $t_1 - x_1/a_1 > t_2$ . Consider a contract  $C' = ((t_2 + x_1/a_1, x_2), (t_2, 0))$ . Then

$$\mu(\Theta(C')) t_2 \leq \bar{U}_2 = \mu(\Theta(C)) t_2,$$

where the first inequality follows from optimal search of type 2 workers for  $C'$  and the second equality from optimal search of the same workers for  $C$  together with the assumption that  $p_2(C) = \pi_2 > 0$ . This implies that  $\Theta(C') \leq \Theta(C)$ . Hence,

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) t_2,$$

where the first equality follows from optimal search of type 1 workers for  $C$  and the assumption that  $p_1(C) = \pi_1 > 0$  and the second inequality comes from  $\Theta(C') \leq \Theta(C)$  and the assumption that  $t_1 - x_1/a_1 > t_2$ . Hence, it must be that  $p_1(C') = 0$ , and, given that  $\Theta(C') \leq \Theta(C) < \infty$ ,  $p_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . It follows that the expected profits for a firm offering  $C'$  are

$$\eta(\Theta(C'))(b_2 - t_2) > \eta(\Theta(C))(\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k,$$

given that  $\Theta(C') = \Theta(C)$  and  $b_2 - t_2 > b_1 - (t_2 + x_1/a_1) > b_1 - t_1$ , where the last inequality follows from assumption. This shows that  $C'$  represents a profitable deviation, a contradiction. ■

## 6 Insurance

### 6.1 Setup

Our second example is closer to the original Rothschild and Stiglitz (1976) environment, where risk neutral principals offer insurance contracts to risk averse agents who are heterogeneous in their probability of experiencing a loss. This example illustrates several features of our environment. We do not require that utility is quasi-linear. Nor do we require search frictions, but can instead allow the short side of the market to match with certainty. Finally, we show that even when a pooling allocation does not Pareto dominate the equilibrium, a partial pooling allocation, where only some types of agents are pooled together, may be Pareto superior.

To be concrete, we again imagine worker-firm matches, where the productivity of a match is initially unknown. Some workers (agents) are more likely to be productive than others, but firms (principals) can only verify the ex post realization of productivity, not a worker's type. More precisely, a type  $i$  worker produces 1 unit of output with probability  $p_i$  and 0 otherwise. A contract specifies the worker's consumption conditional on whether the pair can produce. Workers are risk averse and firms risk neutral. In the absence of adverse selection, the

marginal utility of consumption would be equalized across states. By incompletely insuring workers against the risk of being unproductive, a firm can keep undesirable workers from directing their search toward a particular contract. We believe that this model may provide an explanation for why firms do not insure workers against layoff risk.

To illustrate that search frictions are not essential for our results, assume that the number of matches is determined by the short side of the market,  $\mu(\theta) = \min\{\theta, 1\}$ . This assumption allows us to focus on the risk of matches turning out to be unproductive. It also simplifies our algebra considerably.

Now an action profile consists of a pair of consumption levels,  $y = \{c_e, c_u\}$ . The payoff of a matched type  $i$  worker who undertakes action profile  $\{c_e, c_u\}$  is

$$u_i(c_e, c_u) = p_i U(c_e) + (1 - p_i)U(c_u),$$

where  $p_1 < p_2 < \dots < p_I < 1$  and  $U : [\underline{c}, \infty) \rightarrow \mathbb{R}$  is increasing and strictly concave with  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  for some  $\underline{c} < 0$  and  $U(0) = 0$ . The payoff of a firm matched with a type  $i$  worker who undertakes action profile  $\{c_e, c_u\}$  is

$$v_i(c_e, c_u) = p_i(1 - c_e) - (1 - p_i)c_u.$$

To ensure that assumption A1 is satisfied, we restrict the set of feasible action profiles to  $\mathbb{Y} = \{(c_e, c_u) | c_u + 1 \geq c_e \geq \underline{c} \text{ and } c_u \geq \underline{c}\}$ . Moreover, since a reduction in  $c_u$  raises  $v_i(y)$  and lowers  $u_i(y)$  and is feasible, A2 is satisfied. The assumption that  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  ensures that action profiles of the form  $\{c_e, \underline{c}\}$  yield negative utility for all types and so are not in  $\bar{\mathbb{Y}}$ . To verify A3, consider an incremental increase in  $c_e$  to  $c_e + dc_e$  and an incremental reduction in  $c_u$  to  $c_u - dc_u$  for some  $dc_e > 0$  and  $dc_u > 0$ . For a type  $i$  worker, this raises utility by approximately  $p_i u'(c_e) dc_e - (1 - p_i) u'(c_u) dc_u$ , which is positive if and only if

$$\frac{dc_e}{dc_u} > \frac{1 - p_i}{p_i} \frac{u'(c_u)}{u'(c_e)}.$$

Since  $(1 - p_i)/p_i$  is decreasing in  $i$ , an appropriate choice of  $dc_e/dc_u$  yields an increase in utility if and only if  $j \geq i$ , which verifies A3. Our propositions therefore apply.

Finally, assume  $p_1 \leq k < p_I$ , which ensures that there are no gains from employing the lowest type, even in the absence of asymmetric information, but there may be gains from trade for higher types, for example by setting  $c_e = c_u = p_I - k > 0$ . Let  $i^*$  denote the lowest type without gains from trade, so  $p_{i^*} \leq k < p_{i^*+1}$

## 6.2 Competitive Search Equilibrium

We again characterize a competitive search equilibrium using problem (P):

**Result 4** *There exists a competitive search equilibrium where for all  $i \leq i^*$ ,  $\bar{U}_i = 0$ ; and for all  $i > i^*$ ,  $\theta_i = 1$ ,  $\bar{U}_i > 0$ , and  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$  are the unique solution to*

$$p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$$

and  $p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) = p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1})$ ,

where  $c_{e,i^*} = c_{u,i^*} = 0$ .

**Proof.** For  $i \leq i^*$ , consider the problem (P- $i$ ) without the constraint of keeping out lower types. This relaxed problem should yield a higher payoff

$$\bar{U}_i \leq \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_i U(c_e) + (1 - p_i) U(c_u))$$

s.t.  $\min\{1, \theta^{-1}\} (p_i(1 - c_e) - (1 - p_i)c_u) \geq k$ .

At the solution,  $c_{u,i} = c_{e,i} = c_i$  and so this reduces to

$$\bar{U}_i = \max_{\theta \in [0, \infty], c \geq \underline{c}} \min\{\theta, 1\} U(c)$$

s.t.  $\min\{1, \theta^{-1}\} (p_i - c) \geq k$ .

Either the constraint set is empty (if  $p_i < k + \underline{c}$ ) or there are no points in the constraint set that give positive utility. In any case, this gives  $\bar{U}_i = 0$ .

Turn next to a typical problem (P- $i$ ),  $i > i^*$ :

$$\bar{U}_i = \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_i U(c_e) + (1 - p_i) U(c_u))$$

s.t.  $\min\{1, \theta^{-1}\} (p_i(1 - c_e) - (1 - p_i)c_u) \geq k$

and  $\min\{\theta, 1\} (p_j U(c_e) + (1 - p_j) U(c_u)) \leq \bar{U}_j$  for all  $j < i$ .

We claim first that the solution to this problem sets  $\theta_i = 1$ . If  $\theta_i > 1$ , reducing  $\theta_i$  to 1 relaxes the first constraint without otherwise affecting the solution to the problem. If  $\theta_i < 1$ , consider the following variation: raise  $\theta_i$  to 1 and increase  $c_e$  and reduce  $c_u$  while keeping both  $\theta(p_i U(c_e) + (1 - p_i) U(c_u))$  and  $p_i c_e + (1 - p_i) c_u$  unchanged. That is, the perturbed

consumption levels  $c_e > c_{e,i}$  and  $c_u < c_{u,i}$  are defined by

$$\begin{aligned}\theta_i(p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i})) &= p_i U(c_e) + (1 - p_i)U(c_u) \\ \text{and } p_i c_{e,i} + (1 - p_i)c_{u,i} &= p_i c_e + (1 - p_i)c_u.\end{aligned}$$

By construction, this does not affect the value of the objective function nor the first constraint. Suppose it fails to relax one of the remaining constraints. Then it must be that for some  $j < i$ ,

$$\theta_i(p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i})) \leq p_j U(c_e) + (1 - p_j)U(c_u).$$

Multiply this together with  $p_i U(c_e) + (1 - p_i)U(c_u) = \theta_i(p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i}))$  and simplify to obtain

$$(p_i - p_j)(U(c_{e,i})U(c_u) - U(c_e)U(c_{u,i})) \geq 0.$$

But since  $p_i > p_j$ ,  $c_{e,i} < c_e$ , and  $c_u < c_{u,i}$ , this is a contradiction. Thus the perturbation relaxes each of these constraints. We may therefore without loss of generality focus on the problem with  $\theta_i = 1$ :

$$\begin{aligned}\bar{U}_i &= \max_{(c_e, c_u) \in \mathbb{Y}} (p_i U(c_e) + (1 - p_i)U(c_u)) & \text{(P-}i\text{'}) \\ \text{s.t. } & p_i(1 - c_e) - (1 - p_i)c_u \geq k \\ \text{and } & p_j U(c_e) + (1 - p_j)U(c_u) \leq \bar{U}_j \text{ for all } j < i.\end{aligned}$$

We label this problem for use in the remainder of this proof.

Next we claim that the solution to problem (P- $i'$ ) has  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$ . We proceed by induction. For  $i = i^* + 1$ , we write problem (P- $i'$ ) as

$$\begin{aligned}\bar{U}_i &= \max_{(c_e, c_u) \in \mathbb{Y}} (p_i U(c_e) + (1 - p_i)U(c_u)) \\ \text{s.t. } & p_i(1 - c_e) - (1 - p_i)c_u \geq k \\ \text{and } & p_j U(c_e) + (1 - p_j)U(c_u) \leq 0 \text{ for all } j < i.\end{aligned}$$

$c_{e,i} = c_{u,i} = 0$  satisfies the constraints but leaves the first one slack. Lemma 1 implies that it is possible to do better, i.e. to obtain positive utility. On the other hand, consider any  $(c_{e,i}, c_{u,i})$  that delivers positive utility and satisfies the last constraint, so

$$p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i}) > 0 \text{ and } p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i}) \leq 0.$$

Subtracting inequalities gives

$$(p_i - p_j)(U(c_{e,i}) - U(c_{u,i})) > 0,$$

which proves  $c_{e,i} > c_{u,i}$ . Finally, if  $c_{e,i} > c_{u,i} \geq 0$ , the last constraint is violated, while if  $0 \geq c_{e,i} > c_{u,i}$ , the objective is negative. This proves  $c_{e,i} > 0 > c_{u,i}$  when  $i = i^* + 1$ .

Now suppose we have proven that  $c_{e,j} > c_{e,j-1}$  and  $c_{u,j} < c_{u,j-1}$  for all  $j < i$ ,  $j > i^*$ . Setting  $c_{e,i} = c_{e,i-1}$  and  $c_{u,i} = c_{u,i-1}$  is again feasible because it satisfies all the constraints in problem (P- $i'$ ). This follows because those values satisfy the constraints in problem (P- $(i-1)'$ ) and deliver value  $\bar{U}_{i-1}$ . But this choice is not optimal because it leaves the first constraint slack. So the solution to problem (P- $i'$ ) must be a policy  $(c_{e,i}, c_{u,i})$  that delivers higher utility and satisfies the constraint of excluding type  $i-1$  agents:

$$\begin{aligned} p_i U(c_{e,i}) + (1 - p_i) U(c_{u,i}) &> p_i U(c_{e,i-1}) + (1 - p_i) U(c_{u,i-1}) \\ p_{i-1} U(c_{e,i}) + (1 - p_{i-1}) U(c_{u,i}) &\leq p_{i-1} U(c_{e,i-1}) + (1 - p_{i-1}) U(c_{u,i-1}). \end{aligned}$$

Subtracting inequalities gives

$$(p_i - p_{i-1})(U(c_{e,i}) - U(c_{e,i-1}) - U(c_{u,i}) + U(c_{u,i-1})) > 0.$$

This proves that  $U(c_{e,i}) - U(c_{e,i-1}) > U(c_{u,i}) - U(c_{u,i-1})$ . Finally, if  $U(c_{e,i}) > U(c_{e,i-1})$  and  $U(c_{u,i}) > U(c_{u,i-1})$ , the constraint of excluding type  $i-1$  agents is violated. If  $U(c_{e,i}) < U(c_{e,i-1})$  and  $U(c_{u,i}) < U(c_{u,i-1})$ , the objective function in problem (P- $i'$ ) is reduced compared to setting  $c_{e,i} = c_{e,i-1}$  and  $c_{u,i} = c_{u,i-1}$ . Thus the solution to problem (P- $i'$ ) must have  $U(c_{e,i}) - U(c_{e,i-1}) > 0 > U(c_{u,i}) - U(c_{u,i-1})$ .

Finally, we claim that for all  $i > i^*$ , the equilibrium consumption specified by type  $i$  contracts satisfies two binding constraints, firms earn zero profits and type  $i-1$  workers are indifferent about applying for type  $i$  contracts,

$$\begin{aligned} p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} &= k \\ \text{and } p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) &= \bar{U}_{i-1}. \end{aligned}$$

In particular, these equations will typically exhibit two solutions; we look for the unique solution with  $c_{e,i} > 0 > c_{u,i}$ . It is straightforward to prove that these equations must bind for  $i = i^* + 1$ . For any  $i > i^* + 1$ , suppose we have established this result for type  $i-1$  and

proceed by induction. In particular, we know that

$$\begin{aligned} p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}) &= \bar{U}_{i-1}, \\ p_{i-2}U(c_{e,i-1}) + (1 - p_{i-2})U(c_{u,i-1}) &= \bar{U}_{i-2}. \end{aligned}$$

The first equation comes from the construction of  $\bar{U}_{i-1}$ , the second from the assumption that type  $i - 2$  workers are indifferent about applying for type  $i - 1$  contracts. Suppose that for some  $j < i - 2$

$$p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i}) = \bar{U}_j.$$

Substituting for  $\bar{U}_j$  using the optimal search condition for  $j < i - 2$  gives

$$p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i}) \geq p_j U(c_{e,i-1}) + (1 - p_j)U(c_{u,i-1}).$$

Combining this last inequality with the constraint for type  $i - 1$ ,

$$p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) \leq \bar{U}_{i-1},$$

we obtain  $(p_{i-1} - p_j)(U(c_{e,i-1})U(c_{u,i}) - U(c_{e,i})U(c_{u,i-1})) \geq 0$ . Since  $p_{i-1} > p_j$  for all  $j < i - 2$ , and given that  $c_{e,i} > c_{e,i-1}$ ,  $c_{u,i} < c_{u,i-1}$ , we obtain a contradiction. Instead, the constraint for type  $i - 1$  must bind and the remaining constraints are slack, completing the proof. ■

An interesting feature of a competitive search equilibrium is that  $c_{u,i}$  is negative, so a worker is worse off in a bad match than she would have been without a match at all. If one thinks of a bad match as a layoff, an optimal contract must give a worker less utility if she is laid off than if she never gets a job in order to keep out workers with still lower expected utility.

### 6.3 Pareto Optimality

As in the first example, the equilibrium need not be efficient. Observe that a worker with  $p_i$  close to 1 suffers little from the distortions introduced by adverse selection. At the extreme, if  $p_I = 1$ , setting  $c_{u,I} = \underline{c}$  excludes all other workers without distorting the type  $I$  contract. More generally, adverse selection has the biggest impact on the utility of workers with an intermediate value of  $p_i$ . A Pareto improvement may therefore require only partial pooling.

To be concrete, suppose  $p_1 = 1/4$ ,  $p_2 = 1/2$ , and  $p_3 = 3/4$  and there are equal numbers of type 1 and type 3 workers, so half of all matches are productive. Also assume  $U(c) = \log(1 + c)$  and  $k = 1/8$ . Then in a competitive search equilibrium,  $c_{e,1} = c_{u,1} = 0.125$



and  $\bar{U}_1 = U(0.125)$ ;  $c_{e,2} = 0.786$ ,  $c_{u,2} = -0.036$ , and  $\bar{U}_2 = U(0.312)$ ; and  $c_{e,3} = 0.858$ ,  $c_{u,3} = -0.073$ , and  $\bar{U}_3 = U(0.561)$ . Pooling all three types, the best incentive-feasible allocation sets  $c_e = c_u = 3/8$  and  $\bar{U}_i = U(3/8)$ , since half of all matches are productive. This reduces the utility of a type 3 agent.

Instead consider an allocation that pools type 1 and type 2 workers. If there are sufficiently few type 1 workers, it is feasible to set consumption at  $c_e = c_u > 0.312$ , delivering utility greater than  $\bar{U}_2$  to both type 1 and 2. For example, suppose  $\pi_1 = \pi_3 = 0.1$  and  $\pi_2 = 0.8$ . Then the utility of type 1 and 2 rises to  $U(25/72) = U(0.347)$ . By raising the utility of type 2, it is easier to exclude them from type 3 contracts, reducing the requisite inefficiency of those contracts. This raises the utility of those workers, in this case to  $U(0.573)$ .

## 6.4 Relationship with Rothschild-Stiglitz

This example is similar to the model of Rothschild and Stiglitz (1976, p. 630), where they “consider an individual who will have income of size  $W$  if he is lucky enough to avoid accident. In the event an accident occurs, his income will be only  $W - d$ . The individual can insure against this accident by paying to an insurance company a premium  $\alpha_1$  in return for which he will be paid  $\hat{\alpha}_2$  if an accident occurs. Without insurance his income in the two states, ‘accident,’ ‘no accident,’ was  $(W, W - d)$ ; with insurance it is now  $(W - \alpha_1, W - d + \alpha_2)$  where  $\alpha_2 = \hat{\alpha}_2 - \alpha_1$ .” We can always normalize the utility of an uninsured individual to zero and then express the utility of an individual who anticipates an accident with probability  $p_i$  as

$$u_i(\alpha_1, \alpha_2) = p_i U(W - \alpha_1) + (1 - p_i) U(W - d + \alpha_2) - \kappa_i,$$

where  $\kappa_i \equiv p_i U(W) + (1 - p_i) U(W - d)$ . Setting  $W = d = 1$  and defining  $c_e = 1 - \alpha_1$  and  $c_u = \alpha_2$ , this is equivalent to our example, except for a level shift in the utility function. Still, our characterization carries through to this environment, with one notable exception: the cost of posting contracts may prevent trade for the highest types as well. For example, if  $U(p_i - k) < p_i U(1) + (1 - p_i) U(0)$ , as may happen for  $p_i$  close to 1, there gains from insurance do not cover the contract posting cost.

Rothschild and Stiglitz (1976) prove that in any equilibrium, principals who attract type  $i$  agents,  $i > 1$ , offer incomplete insurance so as to deter type  $i - 1$  agents. Under some conditions, however, such an equilibrium might not exist. Starting from this configuration of contracts, a principal may consider deviating by offering a pooling contract that attracts multiple types of agents. This is profitable if the least cost separating contract is Pareto inefficient.

Such a deviation is never profitable in our environment. In Rothschild and Stiglitz (1976), a deviating principal can attract and serve all the agents in the economy, or at least a representative cross-section. In our model a principal cannot serve all the agents who are potentially attracted to a contract. Instead, agents are rationed through the endogenous movement in market tightness  $\theta$ . Whether such a deviation is profitable depends on which agents are most willing to accept a decline in market tightness. In this model, high type agents will quickly give up on the pooling contract if it is too crowded with low type agents. Low type agents, who have a lower outside option,  $\bar{U}_{i-1} < \bar{U}_i$ , are more persistent. A principal who tries to offer a pooling contract will end up with a long queue of type 1 agents, the worst possible outcome.

## 7 Asset Markets

### 7.1 Setup

A feature of the previous examples is that market tightness is not distorted:  $\theta_i$  is at the first-best level for any  $i$ . We now consider an example where principals may use tightness to screen out undesirable types. Although all of our results hold more generally, to stress the point, we assume  $\mu(\theta) = \min\{\theta, 1\}$ , so matching is again determined by the short side of the market and  $\bar{\eta} = 1$ . In this case, without any private information,  $\theta$  would typically be equal to 1.

Consider an asset market where sellers (agents) have private information about the value of their asset, as in Akerlof (1970). Although buyers (principals) always value an asset more than the seller does, some sellers' assets are more valuable than others. Market tightness, or probabilistic trading, is a useful screening device since sellers who hold a more valuable asset are more willing to accept a low probability of trade at a given price. Thus, this example shows how an illiquid market may serve as a useful screening device when asset holders have private information about asset values.

We assume that each type  $i$  seller is endowed with one indivisible object, say an apple, of type  $i$ , with value  $a_i^S > 0$  for the seller and  $a_i^B > 0$  for the buyer, both expressed in units of an outside good. The action profile in a contract for type  $i$  sellers consists of a pair  $\{\alpha_i, t_i\}$ , where  $\alpha_i$  is the probability that the seller gives the buyer the apple and  $t_i$  is the expected transfer of the outside good that the buyer gives to the seller.<sup>6</sup> The payoff of a matched type

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<sup>6</sup>We assume that apples are indivisible and interpret  $\alpha$  as the probability of trade. Equivalently, apples may be divisible, with preferences linear in consumption. Then  $\alpha$  may be interpreted as the fraction of fruit traded.

$i$  seller who reports to be a type  $j$  seller is

$$u_i(\alpha_j, t_j) = t_j - \alpha_j a_i^S,$$

while the payoff of a buyer matched with a type  $i$  seller who reports truthfully is

$$v_i(\alpha_i, t_i) = \alpha_i a_i^B - t_i.$$

Note that we have normalized the payoff to zero in the absence of trade.

We set  $I = 2$  and impose a number of restrictions on payoffs. First, both buyers and sellers prefer type 2 apples and both types of sellers like apples:

$$a_2^S > a_1^S > 0 \text{ and } a_2^B > a_1^B > 0.$$

Second, there are gains from trade, including the cost of posting, if the buyer is sure to trade:

$$a_i^S + k < a_i^B \text{ for } i = 1, 2.$$

The available action profiles are  $\mathbb{Y} = [0, 1] \times [0, 1]$ , with  $\bar{\mathbb{Y}}_i = \{(\alpha, t) \in \mathbb{Y} \mid \alpha a_i^S \leq t \leq \alpha a_i^B - k\}$ . Using these restrictions, we verify our four assumptions. As a preliminary step, note that  $(\alpha, t) \in \bar{\mathbb{Y}}_i$  implies  $\alpha \geq k/(a_i^B - a_i^S) > 0$  and  $t \geq k a_i^S/(a_i^B - a_i^S) > 0$ , so in any equilibrium contract, trades are bounded away from zero. If  $\alpha$  were too close to 0, buyers would be unwilling to post contracts. But then  $t$  must be large enough for sellers to be willing to trade.

Since  $\alpha > 0$  whenever  $(\alpha, t) \in \bar{\mathbb{Y}}_i$ , the restriction  $a_1^B < a_2^B$  ensures A1 holds. A2 holds because for any  $(\alpha, t) \in \bar{\mathbb{Y}}_i$ , a movement to  $(\alpha, t - \varepsilon)$  with  $\varepsilon > 0$  is feasible and raises the buyer's utility. The important assumption is again A3, here guaranteed by the restriction that  $a_1^S < a_2^S$ . Fix  $(\alpha, t) \in \bar{\mathbb{Y}}$  and  $\gamma \in (a_1^S, a_2^S)$ . For arbitrary  $\delta > 0$ , consider the action profile  $(\alpha', t') = (\alpha - \delta, t - \gamma\delta)$ . Such an action profile is feasible for sufficiently small  $\delta$  because  $(\alpha, t) \in \bar{\mathbb{Y}}$  guarantees that  $\alpha > 0$  and  $t > 0$ . By construction,

$$u_2(\alpha', t') - u_2(\alpha, t) = \delta(a_2^S - \gamma) > 0$$

and

$$u_1(\alpha', t') - u_1(\alpha, t) = \delta(a_1^S - \gamma) < 0.$$

Now for fixed  $\varepsilon > 0$ , choose  $\delta \leq \varepsilon/\sqrt{1 + \gamma^2}$ . This ensures  $(\alpha', t') \in B_\varepsilon(\alpha, t)$  and so assumption A3 holds.

## 7.2 Competitive Search Equilibrium

We use problem (P) to characterize the equilibrium.

**Result 5** *There exists a unique competitive search equilibrium with  $\alpha_i = 1$ ,  $t_i = a_i^B - k$ ,  $\theta_1 = 1$ ,  $\bar{U}_1 = a_1^B - a_1^S - k$ ,*

$$\theta_2 = \frac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1, \quad \text{and } \bar{U}_2 = \theta_2(a_2^B - a_2^S - k).$$

**Proof.** Write problem (P-1) as

$$\begin{aligned} \bar{U}_1 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\}(t - \alpha a_1^S) \\ \text{s.t. } &\min\{1, \theta^{-1}\}(\alpha a_1^B - t) \geq k. \end{aligned}$$

Lemma 1 ensures that the constraint is binding, so that we can use it to eliminate  $t$  and rewrite the problem as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\}\alpha(a_1^B - a_1^S) - \theta k.$$

Since  $a_1^B > a_1^S + k$ , it is optimal to set  $\alpha = \theta = 1$ . It follows that  $\bar{U}_1 = a_1^B - a_1^S - k$ .

Next we turn to problem (P-2):

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\}(t - \alpha a_2^S) \\ \text{s.t. } &\min\{1, \theta^{-1}\}(\alpha a_2^B - t) \geq k \\ &\min\{\theta, 1\}(t - \alpha a_1^S) \leq a_1^B - a_1^S - k. \end{aligned}$$

One can again prove that both constraints are binding. Eliminating  $t$  from the problem using the first constraint gives

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\}\alpha(a_2^B - a_2^S) - \theta k \\ \text{s.t. } &\min\{\theta, 1\}\alpha(a_2^B - a_1^S) - \theta k = a_1^B - a_1^S - k. \end{aligned}$$

Use the last constraint to eliminate  $\alpha$ :

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty]} \frac{a_1^B - a_1^S - (1 - \theta)k}{a_2^B - a_1^S} (a_2^B - a_2^S) - \theta k \\ \text{s.t. } &\frac{a_1^B - a_1^S - (1 - \theta)k}{\min\{\theta, 1\}(a_2^B - a_1^S)} \in [0, 1]. \end{aligned}$$

We include the last constraint to remember that  $\alpha$  is a probability, lying between 0 and 1. Since by assumption  $a_1^S < a_2^S < a_2^B$ , the objective function is decreasing in  $\theta$ . We thus set  $\theta$  equal to the smallest value consistent with the two constraints, that is

$$\theta_2 = \frac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1.$$

This implies  $\alpha_2 = 1$ , so the constraint binds. The value of the program,  $\bar{U}_2$ , is then easy to compute. ■

In the absence of private information, we would have  $\theta_2 = 1$  and  $\bar{U}_2 = a_2^B - a_2^S - k$ . Relative to this benchmark, buyers post too few contracts designed to attract type 2 sellers, so that some of them fail to match. Since type 2 sellers hold better apples than type 1 sellers, they are more willing to accept a lower matching probability in return for bigger transfers when they do match. Note that the obvious alternative, setting  $\theta_2 = 1$  but rationing through the probability of exchange,  $\alpha_2 < 1$ , is more costly because it involves creating more contracts at cost  $k$  per contract. Reducing the meeting rate is a more cost-effective rationing mechanism than directly rationing trades in meetings.

### 7.3 Pareto Optimality

We again ask whether there is a feasible allocation that Pareto dominates the equilibrium. Consider the allocation in which only a pooling contract is posted, with  $\alpha_1 = \alpha_2 = 1$  and  $t_1 = t_2 = t$ . That is,  $\bar{Y} = \{C\}$ , where  $C = ((1, t), (1, t))$ . Moreover,  $\tilde{\Theta}(C) = 1$ ,  $\tilde{p}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1$ . Finally, set  $t = \pi_1 a_1^B + \pi_2 a_2^B - k$ . The choice of  $t$  ensures that the resource constraint holds and the choice of  $\lambda$  ensures that markets clear. All the sellers apply to the same contract so they receive trivially their maximum possible utility. Hence, the allocation is incentive feasible. The expected payoff for type  $i$  sellers is

$$\bar{U}_i = \pi_1 a_1^B + \pi_2 a_2^B - a_i^S - k,$$

for  $i = 1, 2$ . Since  $a_1^B < a_2^B$ , type 1 sellers are always better off than in equilibrium. Type 2 sellers are better off if and only if

$$\pi_1 a_1^B + \pi_2 a_2^B - a_2^S - k > \frac{(a_2^B - a_2^S - k)(a_1^B - a_1^S - k)}{a_2^B - a_1^S - k}.$$

Since  $\pi_2 = 1 - \pi_1$ , this reduces to

$$\pi_1 < \frac{a_2^B - a_2^S - k}{a_2^B - a_1^S - k} = \frac{\bar{U}_2}{\bar{U}_1}.$$

By assumption both the numerator and denominator are positive, but the numerator is smaller (the equilibrium gains from trade are smaller for type 2 sellers) because  $a_2^S > a_1^S$ . Thus type 2 sellers prefer the pooling allocation only if there is sufficiently little cross subsidization, so  $\pi_1$  is small. The cost of cross subsidizing type 1 sellers then does not offset the benefit, the increased efficiency of trade.

## 7.4 No Trade

So far, we have assumed that there are gains from trade for both types of apples. This section shows that if there are no gains from trade for type 1 apples,  $a_1^B \leq a_1^S + k$ , the market for type 2 apples shuts down, even if there is still gains from trade,  $a_2^B > a_2^S + k$ . Intuitively, it is only possible to keep agents holding bad apples out of the market by reducing the probability of trade in good apples. But if there is no market in bad apples, agents holding them will be willing to accept an arbitrarily small probability of trade in the good apple market, shutting it down.

**Result 6** *In any competitive search equilibrium,  $\bar{U}_1 = \bar{U}_2 = 0$ .*

**Proof.** Write problem (P-1) as

$$\begin{aligned} \bar{U}_1 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\} (t - \alpha a_1^S) \\ \text{s.t. } &\min\{1, \theta^{-1}\} (\alpha a_1^B - t) \geq k. \end{aligned}$$

Lemma 1 ensures that the constraint is binding, so that we can use it to eliminate  $t$  and rewrite the problem as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\} \alpha (a_1^B - a_1^S) - \theta k.$$

Since  $a_1^B \leq a_1^S + k$ , the maximized value is  $\bar{U}_1 = 0$ , attained by setting  $\theta = 0$ .

Next we turn to problem (P-2):

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], (\alpha, t) \in \mathbb{Y}} \min\{\theta, 1\} (t - \alpha a_2^S) \\ \text{s.t. } & \min\{1, \theta^{-1}\} (\alpha a_2^B - t) \geq k \\ & \min\{\theta, 1\} (t - \alpha a_1^S) \leq 0.\end{aligned}$$

One can again prove that both constraints are binding. Eliminating  $t$  from the problem using the first constraint gives

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\} \alpha (a_2^B - a_2^S) - \theta k \\ \text{s.t. } & \min\{\theta, 1\} \alpha (a_2^B - a_1^S) - \theta k = 0.\end{aligned}$$

Use the last constraint to eliminate  $\alpha$ :

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty]} \frac{a_1^S - a_2^S}{a_2^B - a_1^S} \theta k \\ \text{s.t. } & \frac{\theta k}{\min\{\theta, 1\} (a_2^B - a_1^S)} \in [0, 1].\end{aligned}$$

One can verify that the constraint set is nonempty. But since  $a_2^B > a_2^S > a_1^S$ , the fraction in the objective function is negative and so the optimum is attained by setting  $\theta = 0$ , giving  $\bar{U}_2 = 0$ . ■

## 8 Conclusion

This paper has developed a canonical model of adverse selection in a competitive search equilibrium. Under a version of a single crossing property, we prove that there is a unique equilibrium in which principals offer separating contracts to agents. We characterize the equilibrium via the solution to a set of constrained optimization problems and illustrate the use of our framework through three examples, including versions of the Akerlof (1976) rat race model, the Rothschild and Stiglitz (1976) insurance model, and a simple model of asset trade.

Given the tractability of our framework, we anticipate little difficulty in extending the results to a dynamic environment with repeated rounds of contract posting and search. This is important for many applications. For example, a worker who fails to find a job today may search again the following period.

It may also be interesting to study a framework where the informed party posts contracts. In a standard competitive search model, the equilibrium allocation does not depend on who posts contracts and who searches. With asymmetric information, contract posting by informed parties may introduce multiplicity of equilibrium through the usual signaling mechanism. While the equilibrium we study seems robust to this variant of the model, other equilibria may arise.

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